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Dimensional reduction and moment maps

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Abstract

We give a unified viewpoint of moment maps in the case of symplectic, hyper-Kähler, quaternion-Kähler and holomorphic contact manifolds. The Higgs field can be regarded as a moment map under some additional conditions in each case. Using dimensional reductions and moment maps, we reduce the standard 1 instanton on $\mathbb{H}P^1 \cong S^4$ to an $SO(3)$ instanton on $\mathbb{C}P^1 \times \mathbb{C}P^1$ and the standard 1 instanton on $\mathbb{H}P^n$ to the standard 1 instanton on $Gr_2(\mathbb{C}^{n+1})$.

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1. Introduction

The purpose of this paper is to show that the standard 1 instanton on $\mathbb{H}P^1 \cong S^4$ can be reduced to an $SO(3)$ instanton on $\mathbb{C}P^1 \times \mathbb{C}P^1$ and the standard 1 instanton on $\mathbb{H}P^n$ can be reduced to the standard 1 instanton on $Gr_2(\mathbb{C}^{n+2})$ by dimensional reductions and moment maps (Theorems 4.4 and 4.5). We hope that this method would be useful for finding a quaternion ASD connection, because known examples of quaternion ASD connections are quite a few.

Though the idea of dimensional reduction is well developed in [7,10], we review it in more geometrical way. In Section 2, the Lie derivative and the Higgs field are defined from the viewpoint of a principal fibre bundle. We review the formulae about the Higgs field and these formulae are exploited throughout this paper. In particular, one of the equations in

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the formulae (Proposition 2.10) can be considered as the defining equation of a moment map in the case of symplectic, hyper-Kähler, quaternion-Kähler and holomorphic contact manifolds under some additional conditions. As a result, we obtain the unified viewpoint of these moment maps in Section 3.

For a moment map on a symplectic manifold, this is a well-known fact in the theory of geometric quantization [4]. When we apply this viewpoint in the case of hyper-Kähler manifold, it naturally leads to a conception of an SD connection which is a generalization of a self-dual connection in four-dimensional Riemannian geometry. However, it turns out that an SD connection is a strictly limited object in the higher-dimensional case. In fact, we have obtained Theorem 3.4.

As a result, if the structure group is compact, the Higgs field equals the usual hyper-Kähler moment map [6] under some conditions. On a quaternion-Kähler manifold, Galicki defines the moment map [2]. A quaternion-Kähler manifold has no symplectic form, and so a quaternion-Kähler moment map may look different from symplectic and hyper-Kähler moment maps. Indeed, a quaternion-Kähler moment map is really a section of a vector bundle. However, our viewpoint enables us to give a similar explanation of a quaternion-Kähler moment map. In this case, we also need an SD connection which is defined in the almost same way as an SD connection on a hyper-Kähler manifold. On a quaternion Kähler manifold, an SD connection is a rigid object [11] (see Theorem 3.9). As a result, we obtain the Galicki–Lawson formula [3] (see Proposition 3.10). In the case of a holomorphic contact manifold, a particular connection is also defined using the holomorphic contact form and the holomorphic contact line bundle. The curvature form of the connection is *not* type $(1, 1)$, and so this has no relation with a holomorphic vector bundle structure. But the Higgs field which relates to the connection can be considered as a moment map. In fact, under some conditions, the reduction procedure is possible and the quotient space is a new holomorphic contact manifold (Proposition 3.13).

In the final section, we prove the main theorem. Following the idea of dimensional reduction [10], we pursue it on a Riemannian manifold with an isometry group to obtain a connection on the quotient principal fibre bundle. We shall apply our theory for the principal fibre bundle on the zero momentum level set, and so we need a moment map. To obtain a connection form on the zero momentum level set of S^4, we pull back the standard 1 instanton on S^4. However, in the case of $\mathbb{H}P^n$, we introduce a slightly different way, using the large symmetry of the standard 1 instanton on $\mathbb{H}P^n$. This large symmetry induces the reduction of the structure group.

2. Dimensional reduction

First of all, we review the conception of dimensional reduction from a geometrical viewpoint, because we wish to work mainly on principal fibre bundles (for details, see [10]). In this section, we assume the following assumption.

Assumption 2.1. Let $\pi_V : V \to M$ be a vector bundle over M with structure group G, where G is a compact Lie group. A compact Lie group H acts on M and V from the left in such a way that

- π_V is H-equivariant ($\pi_V(h(v)) = h(\pi_V(v))$ for $v \in V$, $h \in H$), and
- the H action on V commutes with the G action, ($h(vg) = h(v)g$ for $v \in V$, $h \in H$, $g \in G$).

More precisely, H action induces linear actions preserving the G structure on the fibres of V.

Definition 2.2. Let ξ^V and ξ^M be vector fields on V and M, respectively, generated by $\xi \in \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of H.

Remark. By Assumption 2.1, we have $\mathrm{d}\pi_V(\xi^V) = \xi^M$, and we obtain an anti-homomorphism from the Lie algebra $\mathfrak{h}$ to the Lie algebras of vector fields $\mathcal{X}(V)$ or $\mathcal{X}(M)$, respectively.

Lemma 2.3. *Let s be a section of V. When we regard a section s as a map from M to V, the differential of s is denoted by* $\mathrm{d}s$. *Then* $\mathrm{d}s(\xi^M) - \xi^V$ *is a vector field along the fibre on V, where* $\xi \in \mathfrak{h}$.

Proof. From the equivariance of π_V, it follows that $\mathrm{d}\pi_V(\mathrm{d}s(\xi^M) - \xi^V) = \mathrm{d}(\pi_V s)(\xi^M) - \xi^M = 0$. □

Definition 2.4 (Mason and Woodhouse [10, p. 28]). Using a natural identification between the fibre V_x of V and the tangent vector space along the fibre $T_v(V_x)$, where x is a point in M and $v \in V_x$, we can regard $\mathrm{d}s(\xi^M) - \xi^V$ as a section of V, for each $\xi \in \mathfrak{h}$. Then a differential operator $L_\xi : \Gamma(V) \to \Gamma(V)$ is defined as

$$(L_\xi s)_x := \mathrm{d}s_x(\xi^M) - \xi^V_{s(x)},$$

where $\Gamma(V)$ is the space of smooth sections of V. We shall call L_ξ a Lie derivative.

Lemma 2.5 (Mason and Woodhouse [10, p. 28]). *We have the following formulae*:

1. $L_\xi(s_1 + s_2) = L_\xi s_1 + L_\xi s_2$ *for arbitrary* $s_1, s_2 \in \Gamma(V)$.
2. $L_\xi(fs) = \xi^M(f)s + fL_\xi s$ *for arbitrary* $f \in C^\infty(M)$ *and* $s \in \Gamma(V)$.
3. $L_{[\xi,\eta]} = -[L_\xi, L_\eta]$, *where* $\xi, \eta \in \mathfrak{h}$.

Proof. The formulae (1) and (2) are trivial. Since we have an anti-homomorphism $[\xi, \eta]^M = -[\xi^M, \eta^M]$, the minus sign is needed in assumption (3) in Lemma 2.5. □

Example 2.6. If V is a tangent bundle of M, our Lie derivative equals the usual Lie derivative. In such a case, the Lie algebra $\mathfrak{h}$ is the Lie algebra of vector fields itself, and so the map $\mathfrak{h} \to \mathcal{X}(M)$ is a homomorphism. Hence, we have no contradiction in Lemma 2.5(3).

Definition 2.7 (Mason and Woodhouse [10, p. 49]). Let ∇ be a covariant derivative on V. For $\xi \in \mathfrak{h}$, we define $A_\xi : \Gamma(V) \to \Gamma(V)$:

$$A_\xi := \nabla_{\xi^M} - L_\xi.$$

Remark. The standard argument shows that A_ξ is a section of End V the bundle of endomorphisms of V. We call A_ξ a Higgs field.

Next, the Higgs field is reformulated from the viewpoint of a principal fibre bundle. Let $\pi_P : P \to M$ be the associated principal fibre bundle to V with structure group G. The action of H is assumed to be lifted in the same way as in Assumption 2.1. For any $\xi \in \mathfrak{h}$, we denote by ξ^P the vector field on P generated by ξ. When s is a section of V, the corresponding G-equivariant map from P to V_0 is denoted by $\tilde{s}$, where V_0 is a vector space satisfying $P \times_G V_0 = V$. In other words, we have $s(\pi_P(p)) = [p, \tilde{s}(p)]$, where $[p, v_0]$ is an element of V represented by $(p, v_0) \in P \times V_0$.

Lemma 2.8. *The Lie derivative corresponds to the usual derivative of* $\tilde{s}$: $L_\xi s = [p, \mathrm{d}\tilde{s}(\xi^P)]$, *where* $p \in P$ *and* $\xi \in \mathfrak{h}$.

Proof. When the natural projection $P \times V_0 \to V$ is denoted by π_G, we have $\mathrm{d}s(\xi^M) = \mathrm{d}\pi_G(\xi^P, \mathrm{d}\tilde{s}(\xi^P))$ and $\xi^V = \mathrm{d}\pi_G(\xi^P, 0)$. □

We assume that the covariant derivative ∇ on V corresponds to a connection 1-form ω on P. The definition of the Higgs field and Lemma 2.8 give the following formula.

Lemma 2.9. *We have*

$$A_\xi s = [p, \omega_p(\xi^P)\tilde{s}(p)], \quad p \in P.$$

Remark. The commutativity of the actions of H and G implies that $\xi^P_{pg} = R_{g*}\xi^P_p$. Combined with Lemma 2.9, it follows that the Higgs field A_ξ is also regarded as a section of the adjoint vector bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$. On the other hand, since $\omega_{hp}(\xi^P_{hp}) = \omega_p((\mathrm{Ad}(h^{-1})\xi)^P_p)$, we obtain $A_{\mathrm{Ad}(h)\xi} = hA_\xi h^{-1}$.

Among many formulae which the Higgs field satisfies, we need the following proposition.

Proposition 2.10 (Mason and Woodhouse [10, p. 50]). *Let* $R^\nabla \in \Omega^2(\mathrm{End}\, V)$ *be the curvature* 2*-form of* ∇. *If the connection* 1*-form* ω *is invariant under the action of H on P, then for each* $\xi \in \mathfrak{h}$, *the Higgs field* A_ξ *satisfies the equation*

$$\nabla_X A_\xi = R^\nabla(X, \xi^M),$$

where X is a tangent vector of M.

Proof. The tangent vector X is lifted and extended as the horizontal vector field $\tilde{X}$ on the principal fibre bundle P. From Assumption 2.1, we obtain $L_{\xi^P}\omega = 0$, and so the vector field $[\tilde{X}, \xi^P]$ is horizontal. The corresponding curvature form on P is denoted by Ω. Then, we get $\Omega(\tilde{X}, \xi^P) = \tilde{X}\omega(\xi^P) = \nabla_X A_\xi$. □

Proposition 2.11 (Mason and Woodhouse [10, p. 50]). *Under the same notation as in Proposition* 2.10, *we have*

$$R^{\nabla}(\xi^M, \eta^M) = [A_\xi, A_\eta] - A_{[\xi,\eta]}$$

for arbitrary $\xi, \eta \in \mathfrak{h}$.

Proof. Since the connection is invariant under the action of H, $(L_{\xi^P}\omega)(\eta^P) = \xi^P\omega(\eta^P) - \omega([\xi^P, \eta^P]) = 0$. Then, we obtain

$$\Omega(\xi^P, \eta^P) = -\eta^P\omega(\xi^P) + [\omega(\xi^P), \omega(\eta^P)] = -\omega([\xi, \eta]^P) + [\omega(\xi^P), \omega(\eta^P)].$$

We use Lemma 2.9 to get the result. □

3. Moment maps

In this section, it is explained that the Higgs field can be considered as a moment map in the case of symplectic, hyper-Kähler, quaternion-Kähler and holomorphic contact manifolds under some additional conditions. From this point of view, the equation in Proposition 2.10 is regarded as the defining equation of moment maps and from Remark below Lemma 2.9 ($A_{\mathrm{Ad}(h)\xi} = hA_\xi h^{-1}$), the equivariance of moment maps are automatically satisfied.

In the case of symplectic manifold, this is a well-known fact [4, p. 265] in the theory of geometric quantization. In this theory, one of the important things is the existence of a line bundle with a connection of which the curvature form equals the symplectic form up to a constant. Under Assumption 2.1, the Higgs field is a moment map, because the bundle of endomorphisms is a trivial bundle, and so the Higgs field A_ξ is only a function.

3.1. Hyper-Kähler case

Let (M, g, I, J, K) be a hyper-Kähler manifold. In addition to Assumption 2.1, the compact Lie group H is assumed to act on M preserving the hyper-Kähler structure.

To regard the Higgs field as a moment map, we need a vector bundle with a particular connection.

Definition 3.1. Let V be a vector bundle on a hyper-Kähler manifold M with a connection ∇. We shall call ∇ an *SD connection*, if the curvature 2-form is locally expressed as a linear combination of three Kähler forms ω_I, ω_J and ω_K.

Lemma 3.2. *We assume that the real dimension of hyper-Kähler manifold M is greater than or equal to* 8. *Let E be a vector bundle on M and* ∇ *be an SD connection on E. If a vector bundle E-valued* 2*-form S satisfies* $\mathrm{d}^{\nabla}S = 0$, *then S is parallel with respect to the induced connection*. (*The differential operator* d^{∇} *means the covariant exterior derivative*.)

Proof. For brevity, I, J and K are written as I_1, I_2 and I_3, respectively. By the hypothesis, S is locally expressed as $\sum_{\alpha=1}^{3} B_\alpha \otimes \omega_\alpha$, where $B_\alpha (\alpha = 1, 2, 3)$ are local sections of E. Since all the Kähler forms ω_α are closed, $\mathrm{d}^\nabla S = 0$ if and only if $\sum_{\alpha=1}^{3} \nabla B_\alpha \wedge \omega_\alpha = 0$. For an arbitrary tangent vector X, there exists a unit tangent vector Y such that X is perpendicular to Y, IY, JY and KY, because the real dimension is greater than or equal to 8. Then, we have

$$\begin{aligned} 0 &= \left(\sum_{\alpha=1}^{3} \nabla B_\alpha \wedge \omega_\alpha\right)(X, Y, IY) \\ &= \sum_{\alpha=1}^{3} \nabla_X B_\alpha \omega_\alpha(Y, IY) + \nabla_Y B_\alpha \omega_\alpha(IY, X) + \nabla_{IY} B_\alpha \omega_\alpha(X, Y) \\ &= \sum_{\alpha=1}^{3} \nabla_X B_\alpha \omega_\alpha(Y, IY) = \nabla_X B_1. \end{aligned}$$

In a similar way, we obtain $\nabla_X B_2 = 0$ and $\nabla_X B_3 = 0$. Since three Kähler forms are also parallel with respect to the Riemannian connection, it follows that $\nabla S = 0$. □

Corollary 3.3. *If the real dimension of hyper-Kähler manifold M is greater than or equal to* 8, *then the curvature form of an SD connection is parallel with the induced connection, and so an SD connection is a Yang–Mills connection.*

Theorem 3.4. *If the real dimension of a hyper-Kähler manifold M is greater than or equal to* 8, *then the holonomy algebra of an SD connection is commutative.*

Proof. Let V be a vector bundle with an SD connection ∇. By definition, the curvature 2-form R^∇ is locally expressed as $\sum_{\alpha=1}^{3} A_\alpha \otimes \omega_\alpha$, where $A_\alpha (\alpha = 1, 2, 3)$ are local sections of $\mathrm{End}\, V$. Combined with Corollary 3.3, the Ricci identity yields $[R^\nabla, A_\beta] = 0$, and so

$$0 = \left[\sum_{\alpha=1}^{3} A_\alpha \otimes \omega_\alpha, A_\beta\right] = [A_1, A_\beta] \otimes \omega_1 + [A_2, A_\beta] \otimes \omega_2 + [A_3, A_\beta] \otimes \omega_3$$

for an arbitrary $\beta = 1, 2, 3$. Since the curvature form is parallel (Corollary 3.3), the Ambrose and Singer theorem (cf. [8, p. 89]) implies the desired result. □

The structure group G of P is assumed to be compact, and so Theorem 3.4 yields that G is a total group, if P admits an SD connection. Then the Higgs field is decomposed into three Lie algebra $\mathfrak{h}$-valued functions. Therefore, Proposition 2.10 yields the following equation:

$$\mathrm{d}\mu_{\alpha_\xi} = \iota_{\xi^M} \omega_\alpha,$$

where $(\mu_1, \mu_2, \mu_3) : M \to \mathfrak{h}^* \times \mathfrak{h}^* \times \mathfrak{h}^*$ is the Higgs field (up to a constant), $\xi \in \mathfrak{h}$ and μ_{α_ξ} is considered as a function $M \to \mathbb{R} (\alpha = 1, 2, 3)$. These are the defining equations of the well-known hyper-Kähler moment map [6].

We give an example of SD connections. We regard a quaternion vector space $(\mathbb{H}^N, \langle\cdot,\cdot\rangle, I, J, K)$ as a flat hyper-Kähler manifold, where, $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^{4N}$ which is the underlying real vector space of $\mathbb{H}^N$.

We put $G := S^1 \times S^1 \times S^1$ and $P := \mathbb{H}^N \times G$ which is a trivial principal fibre bundle on $\mathbb{H}^N$. Using a section $s : \mathbb{H}^N \to P$ which is defined as $s(x) = (x, 1)$, we describe a connection form on P as an $\mathbb{R}^3 = \mathrm{Lie}\, G$-valued 1-form on $\mathbb{H}^N$.

Definition 3.5. We define a connection form on ω_P as

$$\omega_{P_x}(Y) = \sum_{\alpha=1}^{3} \langle Y, I_\alpha x\rangle e_\alpha, \quad x \in \mathbb{H}^N,\ Y \in T_x\mathbb{H}^N \cong \mathbb{H}^N,$$

where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$.

Lemma 3.6. *The connection ω_P is an SD connection. More precisely, the curvature form R^∇ of ω_P is expressed as*

$$R^\nabla = 2\sum_{\alpha=1}^{3} \omega_\alpha \otimes e_\alpha.$$

It is assumed that a compact Lie group H acts linearly on $\mathbb{H}^N$ preserving the quaternion structure. Then the action of H can be lifted to P as

$$h(x, g) = (hx, g), \quad h \in H,\ x \in \mathbb{H}^N,\ g \in (S^1)^3.$$

These actions of H on $\mathbb{H}^N$ and P satisfy Assumption 2.1.

Lemma 3.7. *The connection ω_P is invariant under the action of H on P. The Higgs field A_ξ $(\xi \in \mathfrak{h})$ is written as*

$$(A_\xi)_x = \left(x, \sum_{\alpha=1}^{3} \langle \xi\cdot x, I_\alpha x\rangle e_\alpha\right),$$

where $\xi \cdot x$ is the resulting vector when ξ acts on x by the induced Lie algebra representation on $\mathbb{H}^N$.

For brevity, we put $A_{\alpha\xi} = \langle \xi\cdot x, I_\alpha x\rangle$.

Example (Donaldson [1]). We put $\mathbb{H}^N := M(k;\mathbb{C}) \times M(k;\mathbb{C}) \times M(r,k;\mathbb{C}) \times M(k,r;\mathbb{C})$, where, e.g. $M(r,k;\mathbb{C})$ is a set of complex $r \times k$ matrices. The quaternion structure is defined as $I(\alpha,\beta,a,b) := (\mathrm{i}\alpha, \mathrm{i}\beta, \mathrm{i}a, \mathrm{i}b)$ and $J(\alpha,\beta,a,b) := (-\beta^*, \alpha^*, b^*, -a^*)$ for $(\alpha,\beta,a,b) \in \mathbb{H}^N$. When we regard $\mathbb{H}^N$ as a complex vector space using the complex structure I, then a Hermitian inner product h is defined as $h((\alpha,\beta,a,b),(\gamma,\delta,c,d)) := \mathrm{trace}(\alpha\gamma^* + \beta\delta^* + c^*a + bd^*)$. Finally, $H = U(k)$ acts on $\mathbb{H}^N$ in such a way that $h(\alpha,\beta,a,b) := (h\alpha h^{-1}, h\beta h^{-1}, ah^{-1}, hb)$. Then the Higgs field is as follows:

$$A_{1\xi} = -\mathrm{i}\,\mathrm{trace}([\alpha, \alpha^*] + [\beta, \beta^*] - a^*a + bb^*)\xi,$$

$$A_{2\xi} + \mathrm{i}A_{3\xi} = -2\,\mathrm{trace}([\alpha, \beta] + ba)\xi.$$

3.2. Quaternion-Kähler case

Let (M, g) be a quaternion-Kähler manifold and $S^2\mathbb{H}$ be the quaternion-Kähler structure bundle. Here, the vector bundle $S^2\mathbb{H}$ is a sub-bundle of the bundle of endomorphisms of the tangent bundle of M and is locally spanned by I, J, K. In addition to Assumption 2.1, the compact Lie group H is assumed to act on M preserving the quaternion-Kähler structure.

Galicki gives a notion of moment map on a quaternion-Kähler manifold.

Definition 3.8 (Galicki [2]). A map (more precisely, section) $\mu : M \to \mathfrak{h}^* \otimes S^2\mathbb{H}$ is called a moment map (section) if μ_ξ satisfies the equation

$$\nabla\mu_\xi = \iota_{\xi^M}\sum_{\alpha=1}^{3}\omega_\alpha \otimes I_\alpha,$$

where $\xi \in \mathfrak{h}$, $\omega_\alpha = g(I_\alpha\cdot, \cdot)$ and μ_ξ is considered as a section of $\mathbb{S}^2\mathbb{H}$.

Definition 3.1 still makes sense in the case of quaternion-Kähler manifold, and so we use the same term SD connection. Then the author showed the following theorem.

Theorem 3.9 (Nagatomo [11]). *We assume that the real dimension of a quaternion-Kähler manifold M is greater than or equal to* 8 *and the scalar curvature is not zero. Then the curvature form of an SD connection is parallel and the holonomy algebra is isomorphic to* $\mathfrak{su}(2)$ *or commutative. Moreover, if M is simply-connected, then any non-flat SD connection is gauge-equivalent to the induced connection of* $S^2\mathbb{H}$ *by the Riemannian connection*.

Remark. In the case of hyper-Kähler manifold, the induced connection on "the quaternion-structure bundle" $S^2\mathbb{H}$ is flat ($\nabla I = \nabla J = \nabla K = 0$). As we saw in Lemma 3.6, we have no "rigidity theorem" for an SD connection on a hyper-Kähler manifold.

From Theorem 3.9, if we wish to regard the Higgs field as a moment map, we must consider the lift of the action to $S^2\mathbb{H}$. However, since we assume that the action preserves the quaternion structure, the existence of such a lift is trivial. Next, the same assumption also assures that the usual Lie derivative L_{ξ^M} for sections of the bundle of endomorphisms of the tangent bundle, where $\xi \in \mathfrak{h}$ can be restricted to sections of the sub-bundle $S^2\mathbb{H}$. Then, by Example 2.6, the usual Lie derivative L_{ξ^M} equals our Lie derivative L_ξ in Definition 2.4. Therefore, we obtain the Galicki–Lawson formula.

Proposition 3.10 (Galicki and Lawson [3]). *Under our notation, quaternion-Kähler moment map* μ *satisfies* $\mu_\xi = \nabla_{\xi^M} - L_{\xi^M}$ (*up to a constant*), *where* ∇ *is the induced connection on* $S^2\mathbb{H}$.

3.3. Holomorphic contact manifold

Let (M, L, α) be a holomorphic contact manifold, where L is a holomorphic line bundle and α is an L-valued holomorphic contact form.

We assume that the compact Lie group H acts holomorphically and equivariantly on M and L and the holomorphic contact form α is also preserved under the action of H. Moreover, it is assumed that H acts linearly on the fibres of L. Then, we require that L has an H-invariant Hermitian metric. To work on principal fibre bundles, we denote by $L^{\times}$ the complement of the zero section in the dual bundle L^* of L and regard $L^{\times}$ as the associated principal $\mathbb{C}^{\times}(=\mathbb{C}\setminus 0)$ bundle of L. Consequently, there exists a holomorphic and horizontal 1-form $\tilde{\alpha}$ on $L^{\times}$ corresponding to α. Note that $R_c^*\tilde{\alpha} = c\tilde{\alpha}$ for $c \in \mathbb{C}^{\times}$, where R_c means the right action of $c \in \mathbb{C}^{\times}$ on $L^{\times}$. Our assumption yields that there exists a principal S^1 bundle $\pi_Q : Q \to M$ which is a sub-bundle of $L^{\times}$, and H acts on Q preserving the reduced Hermitian connection ω_Q. The 1-form α is also restricted to Q, for which we use the same notation.

Using the standard homomorphism $U(1) \cong SO(2) \to SO(3)$, we construct a principal $SO(3)$ bundle $P = Q \times_{SO(2)} SO(3)$. Then the connection form ω_Q is extended as a connection form ω_P^Q on P. We also want to extend $\tilde{\alpha}$ to P as an $\mathfrak{so}(3)$-valued 1-form. To do so, we identify $\mathfrak{so}(3)$ with the imaginary part of $\mathbb{H}$. To put it more accurately, we consider $\mathbb{H}$ as a right $Sp(1)$ module and the isomorphism between $\mathfrak{so}(3)$ and $\mathrm{Im}\,\mathbb{H}$ is given by

$$2\begin{pmatrix} 0 & -\mathrm{Im}\,\beta & \mathrm{Re}\,\beta \\ \mathrm{Im}\,\beta & 0 & -b \\ -\mathrm{Re}\,\beta & b & 0 \end{pmatrix} \cong \mathrm{i}b + \mathrm{j}\beta,$$

where $b \in \mathbb{R}$ and $\beta \in \mathbb{C}$. Note that the Lie bracket of $\mathrm{Im}\,\mathbb{H}$ is given by $[q_1, q_2] = q_2q_1 - q_1q_2$ in our case. The subalgebra $\mathfrak{so}(2) \subset \mathfrak{so}(3)$ is identified with $\{\frac{1}{2}\mathrm{i}b \in \mathrm{Im}\,\mathbb{H} | b \in \mathbb{R}\}$. Then the horizontal $\mathfrak{so}(3)$-valued 1-form $\tilde{\alpha}_P$ is defined as follows:

$$\tilde{\alpha}_P(X_p) = \mathrm{Ad}(g^{-1})\{\tfrac{1}{2}\mathrm{j}\tilde{\alpha}(X_q)\},$$

where X_p is a horizontal tangent vector at p of P with respect to the connection ω_P^Q and satisfies $X_p = R_{g*}X_q$ for $q \in Q \subset P,\ g \in SO(3)$.

We define a new connection form ω_P on P as $\omega_P = \omega_P^Q + \tilde{\alpha}_P$.

Lemma 3.11. *The curvature form Ω_P of ω_P is expressed as*

$$\Omega_P = \Omega_P^Q - \tilde{\alpha}_P \wedge \tilde{\alpha}_P + \mathrm{d}^{\nabla}\tilde{\alpha}_P,$$

where Ω_P^Q is the curvature form of ω_P^Q and d^{∇} is the covariant exterior derivative with respect to ω_P^Q. In particular, at $q \in Q \subset P$, we have

$$\Omega_{Pq} = \frac{1}{2}\left\{\mathrm{i}\left(\mathrm{d}\omega_Q + \frac{\mathrm{i}}{2}\tilde{\alpha} \wedge \bar{\tilde{\alpha}}\right) + \mathrm{j}\mathrm{d}^{\nabla}\tilde{\alpha}\right\}.$$

Proof. This is due to a direct computation. □

The action of H on Q can be extended to P and the extended action of H preserves the connection ω_P^Q and the 1-form $\tilde{\alpha}_P$, and so the connection ω_P. Moreover, the action of H on P satisfies Assumption 2.1. We now describe the Higgs field with respect to the connection ω_P.

Lemma 3.12. *We put* $\mathrm{Ad}_Q(P) = Q \times_{SO(2)} \mathfrak{so}(3)$. *Then the Higgs field* $A_\xi (\xi \in \mathfrak{h})$ *can be considered as a section of* $\mathrm{Ad}_Q(P)$. *More precisely, at* $q \in Q \subset P$, *the Higgs field corresponds to*

$$\tfrac{1}{2}\{\mathrm{i}\omega_Q(\xi^Q) + \mathrm{j}\tilde{\alpha}(\xi^Q)\}.$$

Proof. From Lemma 2.9, A_ξ correspond to $\omega_P(\xi^P)$. By the definition of the action of H on P, we obtain $\xi_q^Q = \xi_q^P$ at $q \in Q \subset P$. The definition of the connection form ω_P yields the result. □

In this setting, we shall call the Higgs field $A_\xi = [q, \frac{1}{2}\{\mathrm{i}\omega_{Qq}(\xi^Q) + \mathrm{j}\tilde{\alpha}_q(\xi^Q)\}]$ a *moment map* for a holomorphic contact manifold.

To consider the reduction procedure by a moment map, we assume the following:

1. The curvature form $\mathrm{d}\omega_Q$ determines a real symplectic structure on M. Since the curvature form $\mathrm{d}\omega_Q$ is of type $(1, 1)$, $g(X, Y) = \mathrm{d}\omega_Q(X, JY)$ is a symmetric tensor, where J is the almost complex structure of M. Then, we require that g is a pseudo-Riemannian metric on M and $g(\xi^M, \xi^M) \neq 0$ for all non-zero $\xi \in \mathfrak{h}$. Moreover, J is parallel with respect to the pseudo-Riemannian connection.
2. Under the usual identification between the real tangent bundle TM and the holomorphic tangent bundle $T^{(1,0)}M$, we have $\mathrm{d}\omega_Q(X, \xi^M) = 0$ for an arbitrary $X \in \mathrm{Ker}\, \mathrm{d}^\nabla\alpha$ and each $\xi \in \mathfrak{h}$.
3. The action of H is free.
4. The Higgs field A_ξ is transverse to the zero section for each $\xi \in \mathfrak{h}$.

Remark. For our purpose, assumptions (1)–(3) may be satisfied on only $A^{-1}(0) \subset M$.

Proposition 3.13. *We put* $M_0 = H \setminus A^{-1}(0)$. *Under the above assumptions,* M_0 *has a holomorphic contact manifold structure.*

Proof. From assumption (4), $A^{-1}(0) \subset M$ is a submanifold and the quotient M_0 is also a manifold by assumption (3). Moreover, the same assumption assures the existence of a complex line bundle L_0 on M_0 with a Hermitian structure which is defined as the quotient of L by H over $A^{-1}(0)$.

By Lemma 3.12, for each $x \in A^{-1}(0)$, we have $\omega_{Qq}(\xi^Q) = 0$ and $\alpha_x(\xi^M) = 0$ for all $\xi \in \mathfrak{h}$, where $q \in Q$ satisfies $\pi_Q(q) = x$. Lemma 3.11 yields that a tangent vector X at x is tangent to $A^{-1}(0)$ if and only if $\mathrm{d}\omega_Q(X, \xi^M) = 0$ and $\mathrm{d}^\nabla\alpha(X, \xi^M) = 0$ for all $\xi \in \mathfrak{h}$. Proposition 2.11 implies that $\mathrm{d}\omega_Q(\xi^M, \eta^M) = 0$ and $\mathrm{d}^\nabla\alpha(\xi^M, \eta^M) = 0$ for all

$\xi, \eta \in \mathfrak{h}$. Hence, $\mathfrak{h}_x^M = \{\xi_x^M | \xi \in \mathfrak{h}\}$ is a subspace of $T_x A^{-1}(0)$. Consequently, we obtain a connection form ω_0 on L_0 and L_0-valued 1-form α_0 such that the pull backs of them equals ω_Q and α, respectively. Then assumption (2) yields $(\mathrm{d}^\nabla \alpha_0)^{2(n-\dim H)} \wedge \alpha_0 \neq 0$, where $\dim_{\mathbb{C}} M = 2n+1$.

Finally, we induce a complex structure on M_0 and a holomorphic vector bundle structure on L_0. Using the pseudo-Rimannian metric, we obtain the orthogonal direct sum $T_x A^{-1}(0) = \mathcal{H}_x \oplus \mathfrak{h}_x^M$, because ξ_x^M is not a null vector for each non-zero $\xi \in \mathfrak{h}$. If $X \in \mathcal{H}_x$, then $\mathrm{d}^\nabla \alpha(JX, \xi^M) = 0$, because α is a holomorphic form, $\mathrm{d}\omega_Q(JX, \xi^M) = -g(X, \xi^M) = 0$ by the definition, and $g(JX, \xi^M) = \mathrm{d}\omega_Q(X, \xi^M) = 0$. Hence, we can introduce an almost complex structure of $\mathcal{H}_x$, and so one on M_0. The integrability of this almost complex structure on M_0 can be shown in a similar way to that in the case of Kähler quotient. Then, the curvature form of the connection ω_0 is of type $(1, 1)$, and so L_0 is a holomorphic line bundle on M_0. Now it is clear that α_0 is a holomorphic contact form. □

4. Reduction of quaternion ASD connection

We review the idea of dimensional reduction [10], mainly to fix notation. Assumption 2.1 is also assumed in this section. However, instead of a vector bundle V, we use the associated principal fibre bundle P. The principal fibre bundle $\pi_P : P \to M$ is assumed to have an invariant connection form ω under the action of the compact Lie group H.

If the action of H on M is free, we obtain the quotient $\pi_0^M : M \to M_0 = H \setminus M$ and the principal fibre bundle $\pi_{P_0} : P_0 = H \setminus P \to M_0$. (The natural projection is denoted by $\pi_0^P : P \to P_0$.) We wish to induce a connection form on the principal fibre bundle P_0. To do so, we assume that M has an invariant Riemannian metric g under the action of H. Since the connection ω gives a splitting of the tangent bundle of P into the horizontal sub-bundle $\mathcal{H}$ and the vector bundle along the fibres, we define a metric $g_{\mathcal{H}}$ on $\mathcal{H}$ induced by g, using the identification $\mathcal{H}_p \cong T_x M$, where $\pi_P(p) = x$. Then $g_{\mathcal{H}}$ is invariant under the action of H, because of Assumption 2.1 and the invariance of g and ω. The subspace of $T_p P$ generated by $\mathfrak{h}$ is denoted by $\mathfrak{h}_p^P$. Using the projection $\pi_{\mathcal{H}} : T_p P \to \mathcal{H}_p$ induced by the connection ω, we obtain $\dim \pi_{\mathcal{H}}(\mathfrak{h}_p^P) = \dim H$, because the action of H is free. If we denote by $\pi_{\mathcal{H}}(\mathfrak{h}_p^P)^\perp (\subset \mathcal{H}_p)$ the orthogonal complement of $\pi_{\mathcal{H}}(\mathfrak{h}_p^P) \subset \mathcal{H}_p$, then the invariance of $g_{\mathcal{H}}$ and ω under the action of H yields that $h_* \pi_{\mathcal{H}}(\mathfrak{h}_p^P)^\perp = \pi_{\mathcal{H}}(\mathfrak{h}_{hp}^P)^\perp$, where $h \in H$. Hence $\mathcal{H}_{0\pi_0^P(p)} = \mathrm{d}\pi_0^P(\pi_{\mathcal{H}}(\mathfrak{h}_p^P)^\perp)$ defines a subspace of $T_{\pi_0^P(p)} P_0$. Since $T_{\pi_0^P(p)} P_0 = \mathcal{H}_{0\pi_0^P(p)} \oplus T_{\pi_0^P(p)}(P_{0\pi_0^M \pi_P(p)})$ and $R_{g*}\mathcal{H}_{0\pi_0^P(p)} = \mathcal{H}_{0\pi_0^P(pg)}$, where $g \in G$, the horizontal distribution $\{\mathcal{H}_{0\pi_0^P(p)} | p \in P\}$ defines a connection form ω_0 on P_0.

By pulling back the connection form ω_0 to P by the map $\pi_0^P : P \to P_0$, we obtain a new connection form $\omega^0 = \pi_0^{P*}\omega_0$. By definition, ω^0 is also invariant under the action of H. Though the Higgs field A_ξ^0 can be considered with respect to the connection ω^0, the definition of the Higgs field implies that $A_\xi^0 = 0$ for all $\xi \in \mathfrak{h}$. Then Propositions 2.10 and 2.11 yield

$$R^{\nabla^0}(X, \xi^M) = 0, \qquad R^{\nabla^0}(\xi^M, \eta^M) = 0 \quad \text{for all } X \in T_x M, \ \xi, \eta \in \mathfrak{h}, \tag{1}$$

where R^{∇^0} is the curvature tensor of ω^0.

We define a $\mathfrak{g}$-valued 1-form on P as $\theta := \omega - \omega^0$ and regard θ as the vector bundle $\mathrm{Ad}(P) := P \times_{\mathrm{Ad}} \mathfrak{g}$-valued 1-form on M. Then, we have

$$\theta(X) = 0 \quad \text{for } \ X \in \mathfrak{h}^{M\perp}, \qquad \theta(\xi^M) = A_\xi \quad \text{for } \ \xi \in \mathfrak{h}, \tag{2}$$

where $\mathfrak{h}^{M\perp}$ is the orthogonal complement of the subspace $\mathfrak{h}^M$ of TM generated by $\mathfrak{h}$. A direct computation shows that

$$R^\nabla = R^{\nabla^0} + \mathrm{d}^{\nabla^0}\theta + \theta \wedge \theta, \tag{3}$$

where R^∇ is the curvature tensor of ω and d^{∇^0} means the covariant exterior derivative with respect to ω^0.

If $X, Y \in \mathfrak{h}^{M\perp}$, we extend X and Y as vector fields $\tilde{X}$ and $\tilde{Y}$ satisfying $\tilde{X}, \tilde{Y} \in \mathfrak{h}^{M\perp}$, respectively. Then, since the action of H preserves the Riemannian metric g, we obtain a skew-symmetric tensor $C : \mathfrak{h}^{M\perp} \times \mathfrak{h}^{M\perp} \to \mathfrak{h}$ such that $C(X, Y)^M$ equals the orthogonal projection of $[\tilde{X}, \tilde{Y}]$ to $\mathfrak{h}^M$.

Lemma 4.1. *For arbitrary $X, Y \in \mathfrak{h}^{M\perp}$, we have*

$$R^\nabla(X, Y) = R^{\nabla^0}(X, Y) - A_{C(X,Y)}.$$

Proof. Eqs. (2) and (3) imply $R^\nabla(X, Y) = R^{\nabla^0}(X, Y) + \mathrm{d}^{\nabla^0}\theta(X, Y)$. Using the extended vector fields $\tilde{X}$ and $\tilde{Y}$ as above and Eq. (2), we get $\mathrm{d}^{\nabla^0}\theta(X, Y) = -\theta([\tilde{X}, \tilde{Y}]) = -A_{C(X,Y)}$. □

Though the following formula is not used in this paper, we formulate it for reader's convenience.

Lemma 4.2. *For arbitrary $X \in \mathfrak{h}^{M\perp}$ and $\xi \in \mathfrak{h}$, we have*

$$\nabla_X A_\xi = \nabla^0_X A_\xi.$$

Proof. Proposition 2.10 and Eqs. (1)–(3) imply $\nabla_X A_\xi = \mathrm{d}^{\nabla^0}\theta(X, \xi^M)$. As usual, X is extended to a vector field $\tilde{X}$ as above. Then, it can be shown that $[\tilde{X}, \xi^M] \in \mathfrak{h}^{M\perp}$ for each $\xi \in \mathfrak{h}$, because ξ^M is a Killing vector field. Using (2), we obtain $\nabla^0_X A_\xi = \nabla^0_{\tilde{X}}(\theta(\xi^M)) = \mathrm{d}^{\nabla^0}\theta(\tilde{X}, \xi^M) + \nabla^0_{\xi^M}(\theta(\tilde{X})) + \theta([\tilde{X}, \xi^M]) = \mathrm{d}^{\nabla^0}\theta(X, \xi^M)$. □

From now on, these formulae are used for us to obtain the main theorem. We give a definition of (quaternion) ASD connection.

Definition 4.3 (cf. Mamone Capria and Salamon [9]). Let M be a quaternion-Kähler manifold and V be a vector bundle with a connection ∇. Then ∇ is called an ASD connection if the curvature tensor R^∇ satisfies

$$R^\nabla(IX, IY) = R^\nabla(JX, JY) = R^\nabla(KX, KY) = R^\nabla(X, Y).$$

Now two examples of ASD connections are given. The quaternion projective space $\mathbb{H}P^n$ is written as $Sp(n+1)/Sp(1) \times Sp(n)$. If we put $P = Sp(n+1)/Sp(1)$, then P is a principal fibre bundle on $\mathbb{H}P^n$ with structure group $Sp(n)$. Then the induced connection on P by the canonical connection is an ASD connection [9]. We call this ASD connection the standard 1 instanton on $\mathbb{H}P^n$. More geometrically, the standard 1 instanton is nothing but the canonical connection on the quotient bundle of a trivial bundle $\mathbb{H}P^n \times \mathbb{C}^{2n+2}$ by the tautological quaternion line bundle.

Next, complex Grassmaniann manifold $Gr_2(\mathbb{C}^{n+2})$ is one of quaternion-Kähler manifolds and is expressed as $SU(n+2)/S(U(2) \times U(n))$. If we put $P' = SU(n+2)/SU(2)$, then P' is a principal fibre bundle on $Gr_2(\mathbb{C}^{n+2})$ with structure group $U(n)$. Then the induced connection on P' by the canonical connection is an ASD connection [12]. We call this ASD connection the standard 1 instanton on $Gr_2(\mathbb{C}^{n+2})$. More geometrically, the standard 1 instanton is nothing but the canonical connection on the quotient bundle of a trivial bundle $Gr_2(\mathbb{C}^{n+2}) \times \mathbb{C}^{n+2}$ by the tautological bundle.

Theorem 4.4. *The standard* 1 *instanton on* $\mathbb{H}P^1 \cong S^4$ *is reduced to an* $SO(3)$ *ASD connection on* $\mathbb{C}P^1 \times \mathbb{C}P^1$ *by the dimensional reduction and the moment map reduction.*

Proof. We pull back the standard 1 instanton on S^4 to obtain a holomorphic vector bundle on $\mathbb{C}P^3$ which is the Penrose twistor space of S^4. An S^1 action on $\mathbb{C}P^3$ is defined as $e^{i\theta}[z_0; z_1; z_2; z_3] := [e^{i\theta}z_0; e^{i\theta}z_1; e^{-i\theta}z_2; e^{-i\theta}z_3]$, where z_0, z_1, z_2 and z_3 are the homogeneous coordinates on $\mathbb{C}P^3$. As a homogeneous space, $\mathbb{C}P^3$ is written as $Sp(2)/U(1) \times Sp(1)$ and $Sp(2)/U(1)$ is a principal $Sp(1)$ bundle on $\mathbb{C}P^3$, which is denoted by P. Then, the pull back connection ω of the standard 1 instanton can be regarded as a connection on P. A subgroup S^1 of $Sp(2)$ is defined as $\mathrm{diag}(e^{i\theta}, e^{i\theta}, e^{-i\theta}, e^{-i\theta})$ in a matrix representation of $Sp(2)$. Then, the S^1 action on $\mathbb{C}P^3$ can be lifted to $Sp(2)$ by the left multiplication of the subgroup S^1. We define $\xi \in \mathrm{Lie}(S^1) \subset \mathfrak{sp}(2)$ as $\xi := \mathrm{diag}(i, i, -i, -i)$. The moment map μ_ξ for the S^1 action is $\mu_\xi([z_0; z_1; z_2; z_3]) = |z_0|^2 + |z_1|^2 - |z_2|^2 - |z_3|^2$. Hence $S(U(2) \times U(2)) \subset SU(4)$ acts on $M := \mu^{-1}(0)$ transitively. The isotoropy subgroup at $[1; 0; 1; 0]$ of $S(U(2) \times U(2))$ is denoted by K. We fix an $SU(4)$-invariant inner product on $\mathfrak{su}(4)$ and restrict it to the subalgebra $\mathrm{Lie}(S(U(2) \times U(2)))$. Then we obtain the orthogonal decomposition $\mathrm{Lie}(S(U(2) \times U(2))) = \mathfrak{k} \oplus \mathbb{R}\xi \oplus \mathfrak{m}_0$, where $\mathfrak{k} := \mathrm{Lie}\, K$ and $\mathfrak{m}_0$ is spanned by matrices

$$\left(\begin{array}{cc|c} 0 & -1 & \multirow{2}{*}{O} \\ 1 & 0 & \\ \hline \multicolumn{2}{c|}{O} & O \end{array}\right), \left(\begin{array}{cc|c} 0 & i & \multirow{2}{*}{O} \\ i & 0 & \\ \hline \multicolumn{2}{c|}{O} & O \end{array}\right), \left(\begin{array}{c|cc} O & \multicolumn{2}{c}{O} \\ \hline \multirow{2}{*}{O} & 0 & -1 \\ & 1 & 0 \end{array}\right), \left(\begin{array}{c|cc} O & \multicolumn{2}{c}{O} \\ \hline \multirow{2}{*}{O} & 0 & i \\ & i & 0 \end{array}\right).$$

If we restrict the inner product on $\mathfrak{su}(4)$ to $\mathbb{R}\xi \oplus \mathfrak{m}_0$, we obtain a left-invariant metric on M and this metric equals the induced metric by the Fubini–Study metric on $\mathbb{C}P^3$. The principal fibre bundle P and the connection ω are pull backed to M and the pull backs are denoted by the same symbols P and ω, respectively. The S^1 actions on M and P satisfy Assumption 2.1 when we put $S^1 = H$, and the connection ω is an invariant connection.

If X_x is a tangent vector at $x \in M$ which is orthogonal to ξ_x^M, then there exists an $X \in \mathfrak{m}$ and an $a \in S(U(2) \times U(2))$ such that $d\pi(L_{a*}X) = X_x$, where $\pi : S(U(2) \times U(2)) \to M$

is a natural projection and L_a is the left translation by a on $S(U(2) \times U(2))$. We put $X_a = \mathrm{Ad}(a)X$. Using our notation, $X_a^M = \mathrm{d}\pi(R_* X_a)$ (R means the right translation) is a vector field on M which equals X_x at $x \in M$. Then X_a^M is orthogonal to ξ^M everywhere on M, because $\mathrm{Ad}(b)\xi = \xi$ for any $b \in S(U(2) \times U(2))$. Therefore, when we compute the tensor $C(X, Y)$, we may use X_a^M and Y_a^M. Since we have $[X_a^M, Y_a^M] = -[X_a, Y_a]^M$ and $[X_a, Y_a]$ is orthogonal to ξ, this implies $C(X, Y) = 0$ for arbitrary $X, Y \in \mathfrak{h}^{M\perp}$, where $S^1 = H$. From Lemma 4.1, we know that the induced connection ω_0 on $P_0 = S^1 \setminus P$ has a curvature form of type (1, 1). Since the S^1 action is not free on M, the structure group of P_0 is isomorphic to $Sp(1)/\mathbb{Z}_2 \cong SO(3)$. Then the first Chern class of P_0 vanishes, and so the curvature form of ω_0 is orthogonal to the Kähler form on $\mathbb{C}P^1 \times \mathbb{C}P^1$. Consequently, the connection ω_0 is an ASD connection. □

Theorem 4.5. *The standard* 1 *instanton on* $\mathbb{H}P^n$ *is reduced to the standard* 1 *instanton on* $Gr_2(\mathbb{C}^{n+1})$ *by the dimensional reduction and the moment map reduction.*

Proof. Galicki [2] shows that $Gr_2(\mathbb{C}^{n+1})$ can be obtained as the moment map reduction from $\mathbb{H}P^n$ by an S^1 action. We regard a quaternion vector space $\mathbb{H}^{n+1}$ as a right $\mathbb{H}$ module. Then, the S^1 action on $\mathbb{H}P^n$ is defined by $\mathrm{e}^{\mathrm{i}\theta}[q_0, q_1, \dots, q_n] := [\mathrm{e}^{\mathrm{i}\theta}q_0, \mathrm{e}^{\mathrm{i}\theta}q_1, \dots, \mathrm{e}^{\mathrm{i}\theta}q_n]$, where $q_0, \dots, q_n$ are homogeneous coordinates on $\mathbb{H}P^n$. This S^1 action on $\mathbb{H}P^n$ can be lifted to $Sp(n+1)$, if we regard $S^1 \cong U(1)$ as the subgroup of $Sp(n+1)$, using the "diagonal" embedding (cf. the proof of Theorem 4.4). This lifted action on $Sp(n+1)$ induces an S^1 action on P. These actions of $U(1)$ on $\mathbb{H}P^n$ and P satisfy Assumption 2.1, when we put $U(1) = H$. Then, the standard 1 instanton is preserved by the S^1 action, because the canonical connection is preserved. If we denote by M the zero momentum level set, then the subgroup $U(n+1)$ of $Sp(n+1)$ acts on M transitively, where $U(n+1)$ is embedded in the standard way. (When we put $q_i = z_i + \mathrm{j}w_i$, where z_i and w_i represent complex numbers, $M = \{[q_0, q_1, \dots, q_n] \in \mathbb{H}P^n | \sum_{i=0}^n |z_i|^2 = \sum_{i=0}^n |w_i|^2,\ \sum_{i=0}^n z_i w_i = 0\}$.) As a homogeneous space, M is written as $U(n+1)/K$, and we decompose the Lie algebra $\mathfrak{u}(n+1)$ into the subspaces $\mathfrak{k}$ and $\mathfrak{m}$, where $\mathfrak{k}$ is the Lie algebra of K, $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ and an inner product on $\mathfrak{u}(n+1)$ is induced by an invariant inner product on $\mathfrak{sp}(n+1)$. The invariant inner product defines a bi-invariant metric on $Sp(n+1)$ and so $Sp(n)$-invariant metric g_P on P. In addition, the left-invariant metric g on M induced by the restricted inner product on $\mathfrak{m}$ equals the induced metric from $\mathbb{H}P^n$. The pull back principal fibre bundle $P|_M$ on M is also denoted by P and the pull back connection of the standard 1 instanton on P is denoted by ω_P.

To put it more accurately, we pick up $g_0 \in Sp(n+1)$ such that

$$g_0 = \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{E_{n+1}}} & \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix} & O \\ \cline{3-4} & & O & -E_{n-1} \\ \hline \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & O & \multicolumn{2}{c}{\multirow{2}{*}{E_{n+1}}} \\ \cline{1-2} O & E_{n-1} & & \end{array} \right).$$

Under the natural projection $\pi : Sp(n+1) \to \mathbb{H}P^n$, g_0 represents a point in M. Using our notation

$$K = \left\{ \left(\begin{array}{c|c} A_1 & O \\ \hline O & A_2 \end{array} \right) \in U(n+1) \middle| A_1 \in SU(2),\ A_2 \in U(n-1) \right\},$$

and

$$\mathfrak{u}(n+1) = \mathfrak{k} \oplus \mathbb{R} \left(\begin{array}{c|c} iE_2 & O \\ \hline O & O \end{array} \right) \oplus \mathfrak{m}_0,$$

where the right-hand side means the orthogonal decomposition and the orthogonal complement of $\mathfrak{k}$ is $\mathfrak{m}$. We regard $\tilde{Q} = U(n+1)g_0$ as a sub-bundle of $Sp(n+1)|_M$. Then $\mathrm{Ad}(g_0^{-1})(K) \cap Sp(1)$ comprises only the unit element. Hence $\tilde{Q}$ can be considered as a sub-bundle of P. Then, using the metric g_P and ω_P, a connection form $\omega_{\tilde{Q}}$ is defined in such a way that the horizontal distribution is the orthogonal projection of the horizontal subspace with respect to ω_P to the tangent space of $\tilde{Q}$. We define a new bundle Q as the quotient of $\tilde{Q}$ by $\mathrm{Ad}(g_0^{-1})(SU(2))$, and so the structure group of Q is isomorphic to $U(n-1)$. The principal fibre bundle Q has a connection form ω_Q inherited by $\omega_{\tilde{Q}}$.

The connection form ω_Q on Q is also considered as follows. The canonical connection on $Sp(n+1)$ and the bi-invariant metric on $Sp(n+1)$ define a connection on $\tilde{Q}$ in a similar way. This is nothing but the Riemannian connection on M. Then the quotient bundle of $\tilde{Q}$ by $\mathrm{Ad}(g^{-1})(SU(2))$ is isomorphic to the bundle Q in a natural way and the inherited connection form equals the connection form ω_Q. Since $S^1 \cong U(1)$ is also a subgroup of $U(n+1)$ from our definition, S^1 acts on Q and this observation yields that ω_Q is invariant under the action of $U(n+1)$ and in particular, the action of S^1. These action on M and Q satisfy Assumption 2.1, when we substitute P and H in Assumption 2.1 into Q and S^1, respectively. As usual, we define a principal fibre bundle $Q_0 = H \setminus Q$ on $M_0 = H \setminus M$ and get a connection form ω_0 from ω_Q.

The definition of ω_0 implies that the horizontal distribution with respect to ω_0 is comprised of the left translation of the subspace $\mathfrak{m}_0$, and so ω_0 is nothing but the connection form induced by the canonical connection on $M_0 = Gr_2(\mathbb{C}^{n+1})$. Therefore, this is the standard 1 instanton on $Gr_2(\mathbb{C}^{n+1})$. □

Remark. In the proof of Theorem 4.5, we used the principal fibre bundle Q. However, we can define a principal fibre bundle $P_0 = H \setminus P$ on M_0 and a connection form on P_0 from ω_P. Then, in a similar way to the proof of Theorem 4.4, the new connection on P_0 is also an ASD connection.

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